Automatic Quantum Error Correction.

Jeff P Barnes and Warren S Warren
Department of Chemistry, Princeton University,
Princeton, NJ 08544-1009

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Abstract

Criteria are given by which dissipative evolution can transfer populations and coherences between quantum subspaces, without a loss of coherence. This results in a form of quantum error correction that is implemented by the joint evolution of a system and a cold bath. It requires no external intervention and, in principal, no ancilla. An example of a system that protects a qubit against spin-flip errors is proposed. It consists of three spin 1/2 magnetic particles, and three modes of a resonator. The qubit is the triple quantum coherence of the spins, and the photons act as ancilla. This article is a greatly expanded version of a letter submitted to *Physical Review Letters*.

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1 Introduction

Quantum computation is of interest because algorithms have been discovered with a significant speed-up over any classical algorithm [1, 2], although these may be unique cases [3]. It is very likely that any physical implementation of a quantum computation will require some form of active quantum error correction. Quantum error correcting codes (QECC) have been devised [1, 4, 5, 6, 7] and experimentally demonstrated [8] that can protect a set of states, the codewords, against a set of errors. QECC is similar in spirit to quantum erasure experiments [9], but with the twist that one is not allowed to manipulate the environment. The surprising fact is that one can still disentangle the codewords from the environment, by transferring the entanglement to another set of states, the ancilla.

However, implementing QECC is a formidable task. There is a high premium placed on using as few qubits (two-level systems) as possible, because as quantum systems grow in size, the number of transitions to be manipulated, unwanted thermal effects [10], and decoherence rates [11] all increase exponentially. But, to take a specific example, the fault-tolerant error detection and repair of even a single qubit can require 15 physical qubits, 5 to store the two codewords, and 10 of which must be in known states of zero entropy [5]. In addition, depending upon how one counts a "logic gate", as many as 28 coherent manipulations of pairs of qubits are required for each repair, because "measuring the stabilizer" means finding the eigenvalues of operators such as $I_{x1}I_{x2}I_{z3}I_{z5}$ (see Fig. (2) of Ref. [5]). Such control over a 32,768-level system is a daunting task, even for a highly coherent spectroscopy such as NMR. Although the efficiency of QECC improves for larger computations, a physical scale-up factor of 22 is still required to factorize a thousand-digit number [12]. Part of the difficulty stems from the need to know which error has struck, in order to repair it. This is because different errors rotate the codeword states about separate axes in Hilbert space. By containing information about which error occurred, the ancilla also provide a conditional axis about which rotation can coherently repair an error. Although this seems like an air-tight argument, there is another way to approach QECC, which we will explore here.

To begin with, note how curious it is that QECC can assign a unique status to the codewords. While a classical probability space inherently contains a privileged basis, Hilbert space does not, and this difference has some striking consequences [13]. In order to function, QECC requires access to ancilla in a state of zero entropy [14], which suggests that one could view QECC as a controlled cooling of the system. It is dissipative evolution that adds classical aspects back into Hilbert space. A large body of work exists that model a diverse range of relaxation phenomena in magnetic [15] and optical resonance [16, 17]. They use Lindblad equations of motion [11]. An earlier approach that did use a Lindblad equation [18] implemented QECC as a limit of very fast external manipulation. In contrast, we seek an approach that is distinct from the concepts of error detection and repair.

We show here that dissipation can be used to implement an "automatic quantum error correction" (AQEC), so called because error correction results exclusively from the joint evolution of a system coupled to a cold, Markovian bath. No intervention is required by the programmer, and in theory, no ancilla are required, although this would be unlikely in practice. Clearly, dissipation can be used to stabilize two distinct states of a quantum system that could store a classical bit of information. What is not obvious, is whether such a system could also hold a qubit, since dissipation usually destroys coherence. The key ideas are to use codewords such that errors must add energy to the system, and to set up the evolution of the system such that excitation and environmental entanglements are expelled from distinct codewords in a symmetric way. This prevents the bath from gaining information on the codewords, and thus coherence can be maintained. In the last section, we outline a system that utilizes three magnetic spin 1/2 particles, and three photons, to implement an AQEC that protects against spin-flip errors. It requires only well understood interactions from magnetic resonance spectroscopy, and is intended to show that AQEC has potentially real-world applications.

2 A simple QECC example.

To begin, let us review the idea of quantum error correction by way of a simple method that protects a single quantum state against environmental entanglements [19]. The idea is similar to that of a quantum eraser experiment [9], but with the twist that one is not allowed to interact the environment. Three two-level systems, labeled as S, A and E, are initially in the state $(a|1_S\rangle + b|0_S\rangle)|0_E\rangle|0_A\rangle$. The goal is to keep S in its current state. An interaction between S and E creates the new state $(a|1_S\rangle|p_E\rangle + b|0_S\rangle|q_E\rangle)|0_A\rangle$. The environment is scattered into two states, $|p_E\rangle$ and $|q_E\rangle$. When $|\langle p_E|q_E\rangle| < 1$, the final state of E depends upon the initial state of S, so they are entangled. If $|p_E\rangle = -|q_E\rangle$, the phase of S has been flipped. To repair S, first note that the entangled state can be written as:

$$\frac{1}{2}\left\{\Big(a|1_S\rangle + b|0_S\rangle\Big)\Big(|p_E\rangle + |q_E\rangle\Big) + \Big(a|1_S\rangle - b|0_S\rangle\Big)\Big(|p_E\rangle - |q_E\rangle\Big)\right\}|0_A\rangle$$

Suppose we can externally manipulate the qubits. Conditionally flip A, if the sign of the state S is flipped from what we expect it to be. In the language of QECC, this is "measuring the stabilizer", or "detecting the error". A serves as the memory. Next, flip the sign of S, conditional on if A detected an error. This is "repairing the state". Both of these actions are unitary transforms on S and A only; the environment is never directly manipulated. After measuring the error, the new state is:

$$\frac{1}{2}\left\{\Big(a|1_S\rangle + b|0_S\rangle\Big)\Big(|p_E\rangle + |q_E\rangle\Big)|0_A\rangle + \Big(a|1_S\rangle - b|0_S\rangle\Big)\Big(|p_E\rangle - |q_E\rangle\Big)|1_A\rangle\right\},$$

and then, after the repair,

$$\Big(a|1_S\rangle + b|0_S\rangle\Big)\frac{1}{2}\left\{\Big(|p_E\rangle + |q_E\rangle\Big)|0_A\rangle + \Big(|p_E\rangle - |q_E\rangle\Big)|1_A\rangle\right\}$$

The original state of S has re-emerged! The entanglement between S and E was transferred to be between A and E, without ever touching E. In order for this scheme to work, it is crucial that A is initially in a single pure state, or in a state of zero entropy. We expect that we can achieve this by cooling A down to 0 °K by the third law of thermodynamics. However, there are systems such as protons in ice or frustrated spin lattices [20] that are postulated to violate the third law. Since cooling these systems still leaves them in a state of non-zero entropy, one should avoid using them as ancilla.

2.1 A dynamical re-formulation of the above example.

The next step is to transform the above error correcting method into the language of a quantum system, relaxing towards equilibrium. We will make use of the operator formalism of NMR [21]. The qubit states are $|0\rangle$ and $|1\rangle$, and the projection operators are $I_\alpha = |1\rangle\langle 1|$ and $I_\beta = |0\rangle\langle 0|$. The raising and lowering operators are $I_+ = |1\rangle\langle 0|$ and $I_- = |0\rangle\langle 1|$, respectively. The Hermitian Pauli operators are $I_x = (I_+ + I_-)/2$, $I_y = (I_+ - I_-)/2i$, and $I_z = (I_\alpha - I_\beta)/2$, and $\vec{I}$ is the vector formed by them. The subscript also indicates which spin is acted upon, so $I_{n,x}$ acts only on spin n. This section is similar to that of Ref. [18], but with the difference that the measurement of the syndrome and the repair process are treated more explicitly.

The Hamiltonian of Eq. (1). is designed to continuously implement the example of the last section. To keep S in the state $|1_S\rangle$, first flip A at the rate d, if S departs from $|1_S\rangle$. If A has flipped, then S is flipped at a rate r. To complete the process, A is cooled at a rate c by interaction with a bath of harmonic oscillators with a broad spectral response. If the bath temperature is low in comparison to the separation of the levels of A, then the density matrix ρ evolves as [11, 16],

$$\begin{aligned} H &= r(I_{A,\beta} + I_{A,\alpha} I_{S,x}) + d(I_{S,\alpha} + I_{S,\beta} I_{A,x}) \\ \frac{\partial \rho}{\partial t} &= -i[H,\rho] - c(I_{A,\alpha}\rho + \rho I_{A,\alpha} - 2I_{A,-}\,\rho\, I_{A,+}) \end{aligned} \tag{1}$$

We suppose the errors occur rapidly in comparison to the system dyanmics, so they are modeled as instantaneous transforms. However, slower interactions can also be corrected [18].

Dissipative evolution is usually handled in Liouville space, where ρ is a vector, and transformations like $-i[H,\rho]$ are matrix-vector multiplications [22]. These matrices are called superoperators, since they operate on operators. Eq. (1) becomes a set of linear differential equations, $\dot{\rho} = \Gamma\rho$. The elements of Γ are then indexed by how they transform the populations and coherences of an orthonormal set that spans the Hilbert space: each element of Γ transforms a $|j\rangle\langle k|$ to a $|n\rangle\langle m|$. When $c = 0$, the evolution is unitary, and Γ has two kinds of eigenvalues: $\lambda = 0$, corresponding to populations of eigenstates of H, $|n\rangle\langle n|$, and $\lambda = \pm i\beta$, corresponding to coherences $|n\rangle\langle m|$ and $|m\rangle\langle n|$. Unfortunately, Liouville space also increases the problem size: N qubits now require an evolution superoperator with 4^N eigenstates.

When evolution is dissipative, Γ is not a symmetric matrix. It can still be written as the outer product of its right and left eigenvectors, $\Gamma = \sum \lambda_n \vec{r}_n \otimes \vec{l}_n$, but in general the $\vec{r}_n$ are not orthogonal. However,

$\vec{l}_n \cdot \vec{r}_m = \delta_{nm}$, which allows one to formally solve the equation of motion for an operator as $\vec{x}(t) = \sum \vec{r}_n(\vec{l}_n \cdot \vec{x}(0)\,)\exp(\lambda_n t)$. The structure of Eq. (1) implies that Γ conserves $\mathrm{tr}(\rho(t)\,)$, but a pure state will not necessarily remain pure.

Does the system of Eq. (1) work? Fig. (1) plots the real parts of the eigenvalues of Γ of Eq. (1) for various values of d, r and c. The only stable states of Γ will be those with $\Re e(\lambda_n) = 0$, and there is only one such state: $|1_S 0_A\rangle\langle 1_S 0_A|$. This is how dissipative evolution can confer a privileged status on a state.

A numerical integration of the system dynamics also shows this. Fig. (2) plots the linear entropy, defined as $0 \le \mathrm{tr}(\rho(t) - \rho^2(t)\,) \le 1$, of $\rho(t)$, starting from the corrupted state $|0_S 0_A\rangle\langle 0_S 0_A|$. The rate at which the error is repaired is dominated by the eigenvalue of Γ with the least negative, but nonzero, real part. Curiously, a larger c is counterproductive, as it traps the state into a cycle:

$$|0_S 0_A\rangle \;\begin{array}{c}\text{detect} \rightarrow \\ \leftarrow \text{cool}\end{array}\; |0_S 1_A\rangle \;\; \text{repair} \rightarrow \;\; |1_S 1_A\rangle \;\; \text{cool} \rightarrow \;\; |1_S 0_A\rangle$$

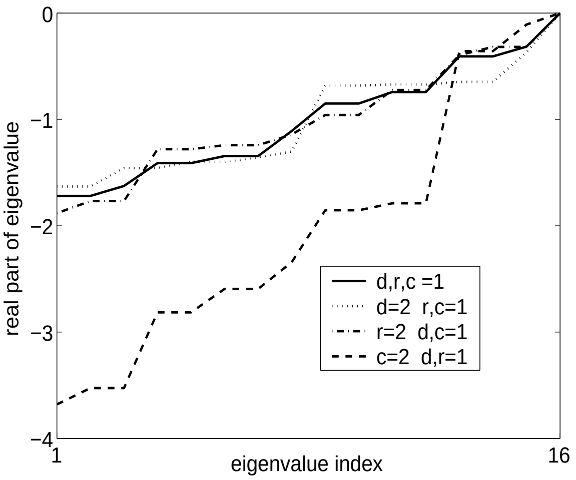


Fig. 1. The real parts of the 16 eigenvalues of the superoperator Γ for some values of d, r and c. Note that there is only a single stable state.

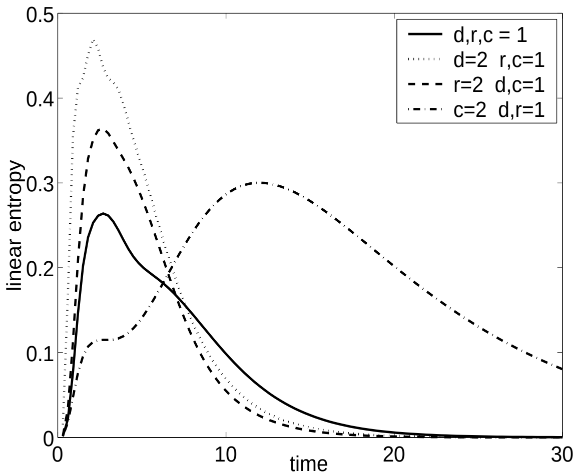


Fig. 2. The linear entropy during the continuous error correction, for d, r, $c = 1$, and doubling each parameter separately. The starting state is $\rho = |0_S 0_A\rangle\langle 0_S 0_A|$.

3 Conditions for AQEC.

3.1 Repairing the Populations.

We now show what conditions are necessary in order that dissipative evolution can automatically protect a subspace of codewords against a given set of errors. We first suppose that the system obeys a Lindblad equation of motion. In general, this is not a trivial assumption [11]. The most speculative condition in deriving a Lindblad equation is that the system and the bath initially factorize. Curiously, it can be justified here on the grounds that a properly working error correction should drive the system to this state. A more troublesome condition is that if degenerate transitions are coupled to the Markov bath, they must couple to orthogonal bath modes [23, 24].

Two conditions can be stated immediately. We are assuming that evolution for a sufficient time, T, can repair any error. Thus, $\exp(\Gamma\, T)$ must be a repair superoperator. Necessary and sufficient conditions for its existence are known from QECC [19]. Under these conditions, the original codeword populations and coherences might be transported elsewhere in Hilbert space, but they are not destroyed. The second condition is that the codewords must be immune from the influence of the bath, or that they form a decoherence free subspace with respect to the system / bath coupling [25].

An example is instructive. Suppose that we are interested in protecting a two-codeword system against spin-flip errors. The system is split into two groups of qubits, S and A, where the A are continuously cooled. This is not necessary for the general argument, which can be formulated entirely in terms of the eigenvalues of Γ. However, it simplifies the physical interpretation. The system evolution is given by [16, 17]

$$\frac{\partial \rho}{\partial t} = -i[\,H, \rho\,] - \sum_{n}^{\text{ancilla}} c_n \Big(I_{n,\alpha}\rho + \rho I_{n,\alpha} - 2 I_{n,-}\rho I_{n,+} \Big) \tag{2}$$

where H acts on both the S and A. The second term irreversibly draws population from the $|1\rangle$ states of the ancilla spins, and places it in the $|0\rangle$ states. Choosing codewords of the form $|\psi_n\rangle|0_A\,\rangle$, for which (1) the A are in their ground states, and (2) the codewords are eigenstates of H, will satisfy the criteria for the decoherence-free subspace.

The need for the QECC conditions can be seen as follows. Suppose we choose the two-S states $|00\rangle$ and $|11\rangle$ as the codewords. But then the errors $I_{1,x}|00\rangle$ and $I_{2,x}|11\rangle$ both result in the same state, $|10\rangle$. Under the Markov approximation, the system can not know which codeword was the original codeword, and so Γ can not repair these errors. But the three-S states $|000\rangle$ and $|111\rangle$ will work, since the spaces spanned by all the errors acting on each codeword are now disjoint. Thus, errors should transfer separate codewords into disjoint subspaces.

Now consider how to repair the codeword populations. The set of errors acting on a codeword, and all the further states that the corrupted codeword evolves into under Γ, form a subspace, as indicated in Fig. (3). Call this subspace the "funnel" associated with the codeword, but excluding the codeword state itself. The name is suggestive of its role in AQEC. The QECC conditions already require the initially excited states to be disjoint between separate codewords. Thus, if we add the third condition that Γ draws all population from each funnel state into its associated codeword, and transfers no amplitude between funnels, then the codeword populations are repaired.

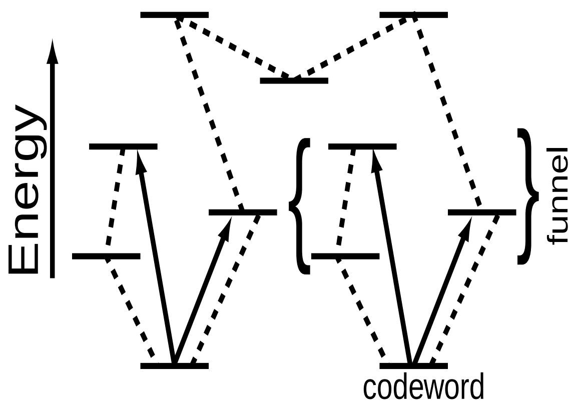


Fig. 3. The level diagram of a hypothetical system. The errors (solid arrows) transfer amplitude into the disjoint funnels associated with each codeword. The three levels between the brackets form the funnel for the labeled codeword to the right. Dissipative cooling of selected transitions (dashed lines) then returns the populations to their original codewords.

An important difference between the usual method by which QECC is implemented, and AQEC, has emerged. By placing the burden of the repair on the system / bath coupling, AQEC, in theory, requires no ancilla. Consider how to repair the error $I_{1,x}$, acting on the three-S and two-A codewords $|000,00\rangle$ and $|111,00\rangle$. Suppose that the unnormalized states $|100,00\rangle \pm |000,10\rangle$ and $|011,00\rangle \pm |111,10\rangle$ are eigenstates of H. The corrupted state $|100,00\rangle$ now periodically becomes the state $|000,10\rangle$, where cooling of the first ancilla returns it to the codeword $|000,00\rangle$. However, we can not choose $|010,00\rangle \pm |000,10\rangle$ and $|101,00\rangle \pm |111,10\rangle$ as eigenstates of H in order to repair the error $I_{2,x}$, because they are not orthogonal to the first set. It seems that we must choose $|010,00\rangle \pm |000,01\rangle$ and $|101,00\rangle \pm |111,01\rangle$ to repair $I_{2,x}$, and $|001,00\rangle \pm |000,11\rangle$ and $|110,00\rangle \pm |111,11\rangle$ to repair $I_{3,x}$. In this strategy, we must be able to distinguish between the different errors in order to be able to repair them, and counting ϵ different errors requires $\log_2\epsilon$ binary digits (or ancilla qubits).

But the third condition for AQEC only requires that H mix the states $|100,00\rangle$, $|010,00\rangle$, and $|001,00\rangle$ with $|000,10\rangle$, even if only partially! Because cooling irreversibly draws probably away from the excited ancilla, any degree of mixing will do. In other words, when the 4×4 block of H corresponding to the above states is diagonalized, each eigenstate should have a non-zero projection onto the state $|000,10\rangle$, and similarly for the other codeword. We still require at least one ancilla here, because the system was split into S and A, and only the A are cooled. If a bath / system coupling is found that directly cools the funnel to codeword transitions as in Fig. (3), then no ancilla are necessary. However, not all the conditions for AQEC have been stated yet. The rest of these come from the seemingly bizarre notion that we can use dissipation to restore a coherence.

3.2 Repairing the Coherences.

The QECC conditions ensure that the codeword coherences are transferred, but not "measured", by the environment. AQEC must transfer them back. What happens to coherences during dissipative evolution is a subtle point, which is best explored by way of a comprehensive example. Codewords of the form $|\psi_n\rangle|00\rangle$, with two ancilla, will serve this purpose. The environment, $|e\rangle$, is initially unentangled with the computer. An interaction, U, can entangle the system so that [26]

$$\begin{aligned} U\Big(a_0|\psi_0\rangle|00\rangle + a_1|\psi_1\rangle|00\rangle + a_2|\psi_2\rangle|00\rangle + a_3|\psi_3\rangle|00\rangle + \cdots\Big)|e\rangle = \\ a_0\Big(u_0^{(0)}|\phi_0^{(0)}\rangle|00\rangle|e_0^{(0)}\rangle + u_0^{(1)}|\phi_0^{(1)}\rangle|00\rangle|e_0^{(1)}\rangle + u_0^{(2)}|\phi_0^{(2)}\rangle|00\rangle|e_0^{(2)}\rangle + \cdots\Big) \\ +a_1\Big(u_1^{(0)}|\phi_1^{(0)}\rangle|00\rangle|e_1^{(0)}\rangle + u_1^{(1)}|\phi_1^{(1)}\rangle|00\rangle|e_1^{(1)}\rangle + u_1^{(2)}|\phi_1^{(2)}\rangle|00\rangle|e_1^{(2)}\rangle + \cdots\Big) \\ +a_2\Big(u_2^{(0)}|\phi_2^{(0)}\rangle|00\rangle|e_2^{(0)}\rangle + u_2^{(1)}|\phi_2^{(1)}\rangle|00\rangle|e_2^{(1)}\rangle + u_2^{(2)}|\phi_2^{(2)}\rangle|00\rangle|e_2^{(2)}\rangle + \cdots\Big) \\ +a_3\Big(u_3^{(0)}|\phi_3^{(0)}\rangle|00\rangle|e_3^{(0)}\rangle + u_3^{(1)}|\phi_3^{(1)}\rangle|00\rangle|e_3^{(1)}\rangle + u_3^{(2)}|\phi_3^{(2)}\rangle|00\rangle|e_3^{(2)}\rangle + \cdots\Big) \end{aligned} \tag{3}$$

After the error, the amplitude originally in each codeword is spread throughout its funnel. While the funnel states $\{|\phi_n^{(k)}\rangle\}$ can be chosen as an orthogonal set for each n, this is not true in general for the $\{|e_n^{(k)}\rangle\}$.

As yet, there is no constraint on either how separate codewords can excite the ancilla qubits, or how the dynamics of the repair should proceed. Suppose H uses the first ancilla to repair the codewords $n = 0$ and 1, the second ancilla to repair n=2, and both ancilla to repair n=3. That is, H mixes each $|\phi_0^{(k)}\rangle|00\rangle$ with $|\psi_0\rangle|10\rangle$, and so on. Let us follow an argument analogous to the "quantum jump" approach [17]. The relaxation process is divided up into small time steps, Δt, during which the system and bath evolve separately. At the end of each interval, a fraction of the amplitude in each excited ancilla state jumps into a de-excited state. Tagging on two more qubits to represent two modes of the cold bath, at the end of each time interval, a fraction of the amplitudes make the following jumps:

$$
\begin{array}{lcl}
|\psi_0\rangle|10\rangle|e_0^{(k)}\rangle|00\rangle & \rightarrow & |\psi_0\rangle|00\rangle|e_0^{(k)}\rangle|10\rangle \\
|\psi_1\rangle|10\rangle|e_1^{(k)}\rangle|00\rangle & \rightarrow & |\psi_1\rangle|00\rangle|e_1^{(k)}\rangle|10\rangle \\
|\psi_2\rangle|01\rangle|e_2^{(k)}\rangle|00\rangle & \rightarrow & |\psi_2\rangle|00\rangle|e_2^{(k)}\rangle|01\rangle \\
|\psi_3\rangle|11\rangle|e_3^{(k)}\rangle|00\rangle & \rightarrow & \left\{ \begin{array}{l} |\psi_3\rangle|10\rangle|e_3^{(k)}\rangle|01\rangle \\ |\psi_3\rangle|01\rangle|e_3^{(k)}\rangle|10\rangle \end{array} \right. \\
|\psi_3\rangle|10\rangle|e_3^{(k)}\rangle|00\rangle & \rightarrow & |\psi_3\rangle|00\rangle|e_3^{(k)}\rangle|10\rangle \\
|\psi_3\rangle|01\rangle|e_3^{(k)}\rangle|00\rangle & \rightarrow & |\psi_3\rangle|00\rangle|e_3^{(k)}\rangle|01\rangle
\end{array}
$$

The entire process is repeated until a time T, when the relaxation process is complete.

The heart of the argument relies on the idea that, in the limit of a large number of cold bath modes interacting with the ancilla, it is very likely that different ancilla that de-excite at different times, will transfer their excitation to orthogonal modes of the bath. Once excited, these modes do not further influence the evolution of the computer, *i.e.* there is no back-reaction from the bath. In this case, after equilibrium is reached, the final wavefunction is given by:

$$
\begin{aligned}
& a_0|\psi_0\rangle|00\rangle u_0^{(0)}|e_0^{(0)}\rangle\left(c_0^{(0)}(\Delta t)|100000\rangle + c_0^{(0)}(2\Delta t)|010000\rangle + c_0^{(0)}(3\Delta t)|001000\rangle + \cdots\right) \\
& +a_0|\psi_0\rangle|00\rangle u_0^{(1)}|e_0^{(1)}\rangle\left(c_0^{(1)}(\Delta t)|100000\rangle + c_0^{(1)}(2\Delta t)|010000\rangle + c_0^{(1)}(3\Delta t)|001000\rangle + \cdots\right) \\
& +a_0|\psi_0\rangle|00\rangle u_0^{(2)}|e_0^{(2)}\rangle\left(c_0^{(2)}(\Delta t)|100000\rangle + c_0^{(2)}(2\Delta t)|010000\rangle + c_0^{(2)}(3\Delta t)|001000\rangle + \cdots\right) \\
& \qquad \cdots \\
& +a_1|\psi_1\rangle|00\rangle u_1^{(0)}|e_1^{(0)}\rangle\left(c_1^{(0)}(\Delta t)|100000\rangle + c_1^{(0)}(2\Delta t)|010000\rangle + c_1^{(0)}(3\Delta t)|001000\rangle + \cdots\right) \\
& +a_1|\psi_1\rangle|00\rangle u_1^{(1)}|e_1^{(1)}\rangle\left(c_1^{(1)}(\Delta t)|100000\rangle + c_1^{(1)}(2\Delta t)|010000\rangle + c_1^{(1)}(3\Delta t)|001000\rangle + \cdots\right) \\
& \qquad \cdots \\
& +a_2|\psi_2\rangle|00\rangle u_2^{(0)}|e_2^{(0)}\rangle\left(c_2^{(0)}(\Delta t)|000100\rangle + c_2^{(0)}(2\Delta t)|000010\rangle + c_2^{(0)}(3\Delta t)|000001\rangle + \cdots\right) \\
& +a_2|\psi_2\rangle|00\rangle u_2^{(1)}|e_2^{(1)}\rangle\left(c_2^{(1)}(\Delta t)|000100\rangle + c_2^{(1)}(2\Delta t)|000010\rangle + c_2^{(1)}(3\Delta t)|000001\rangle + \cdots\right) \\
& \qquad \cdots \\
& +a_3|\psi_3\rangle|00\rangle u_3^{(0)}|e_3^{(0)}\rangle\Big(c_3^{(0)}(\Delta t, 2\Delta t)|100010\rangle + c_3^{(0)}(2\Delta t, \Delta t)|010100\rangle + c_3^{(0)}(\Delta t, 3\Delta t)|010001\rangle \\
& \qquad +c_3^{(0)}(3\Delta t, \Delta t)|001100\rangle + c_3^{(0)}(2\Delta t, 3\Delta t)|010001\rangle + c_3^{(0)}(3\Delta t, 2\Delta t)|001010\rangle + \cdots\Big)
\end{aligned} \tag{4}
$$

For a funnel that uses a single ancilla, the $c_n^{(k)}(m\Delta t)$ are the amplitude to start in the state $|\phi_n^{(k)}\rangle|00\rangle|e_0^{(k)}\rangle$, and transfer an excitation to the bath at $m\Delta t$. Formally, it can be constructed from the system propagator, $\exp(-iHm\Delta t/\hbar)$, and matrix elements of the system / bath interaction. Using more than one excited ancilla results in a two-time dependence for the c. All these functions approach zero for $t \rightarrow T$, due to the irreversible loss of amplitude from the funnel states at earlier times.

The important point is that the $c_n^{(k)}(m\Delta t)$, for different n and m, uniquely label orthogonal modes of the bath. To see the consequences of this, form ρ from Eq. (4) by tracing out the bath and environment. The populations look like this:

$$|\psi_0\rangle|00\rangle\langle\psi_0|\langle 00|\,|a_0|^2 \times \sum_m |u_0^{(0)}c_0^{(0)}(m\Delta t)|e_0^{(0)}\rangle + u_0^{(1)}c_0^{(1)}(m\Delta t)|e_0^{(1)}\rangle + \cdots|^2 \tag{5}$$

with similar expressions for the other codewords. Because of the earlier conditions on Γ, the populations must be repaired (there is no where else for the populations to go). Thus, the sum in Eq. (5) is one. However, from Eq. (4), it is easy to see that the coherence $|\psi_2\rangle\langle\psi_0|$ is zero! Using orthogonal ancilla states between n=0 and 2 resulting in these codewords becoming entangled with orthogonal bath modes. What has happened, is that the pattern of excitation in the bath can be used to determine the probability to be in each codeword. Using orthogonal ancilla leaves a separate pattern of excitation behind, which means the bath has gained information about the system, and coherence is irreversibly lost [11]. Thus, another condition for AQEC is that excitation should be symmetrically removed from separate funnels.

The final condition comes from examining the coherence

$$|\psi_0\rangle|00\rangle\langle\psi_1|\langle 00|\, a_0^\star a_1 \times$$
$$\sum_m \left(u_0^{(0)}c_0^{(0)}(m\Delta t)|e_0^{(0)}\rangle + u_0^{(1)}c_0^{(1)}(m\Delta t)|e_0^{(1)}\rangle + \cdots\right)^\dagger \left(u_1^{(0)}c_1^{(0)}(m\Delta t)|e_1^{(0)}\rangle + u_1^{(1)}c_1^{(1)}(m\Delta t)|e_1^{(1)}\rangle + \cdots\right) \tag{6}$$

The sum is the inner product of two vectors, indexed by m, whose elements are environmental wavefunctions. Each vector individually has a unit norm, so by the Swartz inequality, the sum is one if the inner product of each element is maximum. Thus, the final criteria for AQEC is to have $\sum_k u_n^{(k)}c_n^{(k)}(m\Delta t)|e_n^{(k)}\rangle$ $=\sum_k u_q^{(k)}c_q^{(k)}(m\Delta t)|e_q^{(k)}\rangle$, for each pair of codewords n and q, and at each time $m\Delta t$. Physically, we are again preventing the bath from gaining information about the codewords. In this case, however, the information would be transferred by the pattern of environmental entanglements with the bath, instead of the excitation.

This last requirement is similar to the phase-matching requirement for frequency mixing in non-linear optical materials [27]. Consider the following contrived example. A qubit suffers an error, $\alpha|0\rangle + \beta|1\rangle \rightarrow \alpha|2\rangle + \beta|3\rangle$. These four states have frequencies ω_0, ω_1, ω_2 and ω_3, respectively. The original coherence between $|0\rangle$ and $|1\rangle$ has been transferred to a new set of states. There are no ancilla to this repair; instead, relaxation symmetrically drives $|2\rangle \rightarrow |0\rangle$ and $|3\rangle \rightarrow |1\rangle$ at a steady rate. Therefore, we can write $c_0(t) = \sqrt{\gamma}\exp(-i\{\omega_2 t + \omega_0(T-t)\} - \gamma t/2)$ and $c_1(t) = \sqrt{\gamma}\exp(-i\{\omega_3 t + \omega_1(T-t)\} - \gamma t/2)$. The original coherence gains a factor of $\int c_0^\star(t)c_1(t)dt = \exp(i(\omega_0-\omega_1)T) \times \gamma/(\gamma - i(\omega_0 - \omega_1 - \omega_2 + \omega_3))$. When $\omega_0 - \omega_2 = \omega_1 - \omega_3$, the dynamics between the separate funnels is indistinguishable as far as the bath can discern, and a full repair results.

To summarize, the following are sufficient conditions for AQEC, although they may not all be necessary. In particular, it is likely that the Markov assumption could be relaxed. (**1**) The system obeys a Lindblad equation of motion, with an evolution superoperator Γ [11]. If cooling occurs on degenerate transitions, they must be coupled to orthogonal modes of the bath [23, 24]. (**2**) The eigenstates of H consist of codewords, $|\psi_n\rangle$, the funnel subspaces associated with each codeword, $\{|\phi_n^{(p)}\rangle\}$, and the rest. The codewords obey the conditions of QECC [19, 14], and errors transform codewords only into their associated funnels. If the errors are available as joint system / environmental transforms, U, then check whether $\langle\phi_n^{(k)}|U|\psi_n\rangle = \langle\phi_q^{(k)}|U|\psi_q\rangle$ for all $n \neq q$, and for some labeling, k, of the funnel states. (**3**) The codewords form a decoherence free subspace with respect to the bath [25]. (**4**) Γ does not transfer amplitude between funnels. All funnel populations decay under Γ into their associated codeword populations. (**5**) The dynamics under Γ within each codeword-funnel subspace are identical. If ancilla are used, they must be excited symmetrically between separate codewords. The last conditions, which are the novel aspect of this approach, are necessary in order to repair the coherences of the codewords using dissipation. Alternatively, one could replace the last three conditions with criteria on the eigenvalues and left and right eigenstates of Γ.

4 Some Numerical Simulations of AQEC.

4.1 The Single Codeword Model.

This section provides a better idea of how AQEC works by examining the behavior of a few numerical simulations. Recall the example of keeping S in the state $|1_S\rangle$. We now see that we could have used (the states are ordered as $|0_S 0_A\rangle$, $|0_S 1_A\rangle$, $|1_S 0_A\rangle$ and $|1_S 1_A\rangle$):

$$H/\hbar = \begin{pmatrix} \omega_{00} & 0 & 0 & \mu \\ 0 & \omega_{01} & 0 & 0 \\ 0 & 0 & \omega_{10} & 0 \\ \mu^\star & 0 & 0 & \omega_{11} \end{pmatrix} \quad \text{instead of} \quad H/\hbar = \begin{pmatrix} r & d & 0 & 0 \\ d & 0 & 0 & r \\ 0 & 0 & d & 0 \\ 0 & r & 0 & d \end{pmatrix}. \tag{7}$$

As previously, Γ has one zero eigenvalue, $|1_S 0_A\rangle\langle 1_S 0_A|$. However, the path by which error correction occurs is different: $|0_S 0_A\rangle \leftrightarrow |1_S 1_A\rangle \rightarrow |1_S 0_A\rangle$. This results in a more efficient repair, as seen by comparison of Fig. (2) to Fig. (4). The phase of μ, and the parameters ω_{01} and ω_{10}, are irrelevant, but as $\Delta\omega = \omega_{11} - \omega_{00}$ increases, the first step becomes less efficient. However, it isn't crucial that $\omega_{00} = \omega_{11}$ exactly. The less optimized parameters slow down, but do not halt, the correction process. Note that, in contrast to QECC, a spin-flip error at A is removed without ever influencing S.

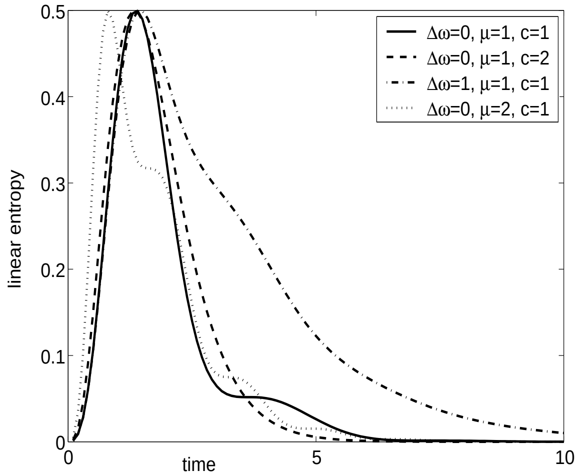


Fig. 4. The linear entropy as a function of time during dissipative AQEC for a single stable state. The starting state is $\rho = |0_S 0_A\rangle\langle 0_S 0_A|$. The more rapidly that H can mix the corrupted state with an excited ancilla, and then cool the ancilla, the more rapid the repair. Non-zero values of $\Delta\omega$, or small values of c, lead to a slower repair.

4.2 The Two-Codeword, Spin-flip Correcting Model.

Let us re-examine the system that protects against spin-flip errors, using three S and two A qubits. The codewords are $|000, 00\rangle$ and $|111, 00\rangle$. Parameterize the system H as shown in Fig. (5):

$$\left(\begin{array}{c|ccc|ccc} \omega_0 & 0 & 0 & 0 & 0 & 0 & 0 \\ \hline 0 & \omega_{e1} & \gamma_{12} & \gamma_{13} & \mu_{11} & \mu_{12} & \mu_{13} \\ 0 & \gamma_{12}^\star & \omega_{e2} & \gamma_{23} & \mu_{21} & \mu_{22} & \mu_{23} \\ 0 & \gamma_{13}^\star & \gamma_{23}^\star & \omega_{e3} & \mu_{31} & \mu_{32} & \mu_{33} \\ \hline 0 & \mu_{11}^\star & \mu_{21}^\star & \mu_{31}^\star & \omega_{c1} & \kappa_{12} & \kappa_{13} \\ 0 & \mu_{12}^\star & \mu_{22}^\star & \mu_{32}^\star & \kappa_{12}^\star & \omega_{c2} & \kappa_{23} \\ 0 & \mu_{13}^\star & \mu_{23}^\star & \mu_{33}^\star & \kappa_{13}^\star & \kappa_{23}^\star & \omega_{c3} \end{array}\right) \begin{array}{l} |000,00\rangle \\ |001,00\rangle \\ |010,00\rangle \\ |100,00\rangle \\ |000,01\rangle \\ |000,10\rangle \\ |000,11\rangle \end{array}$$

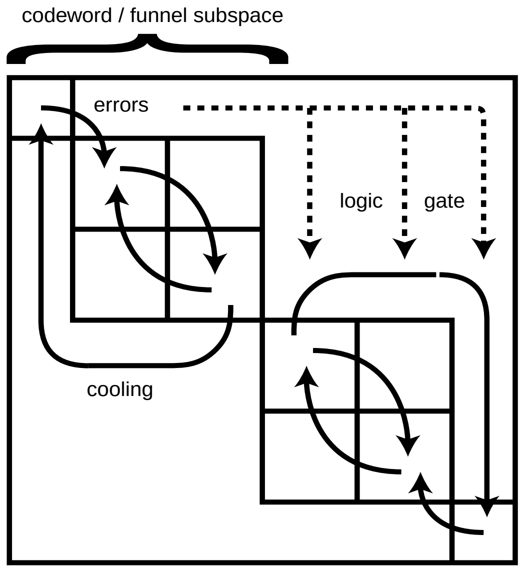


Fig. 5. A parameterized H for a two-codeword AQEC with three S and two A. H is block diagonal, with the two blocks parameterized as shown above (the lines provide a guide for the eye, with a listing of the order of the states for the first funnel / codeword combination). The blocks must be identical, to within a constant offset along the diagonal, so the dynamics between the funnels appears indistinguishable.

The γ mix the different error states, the κ mix the excited ancilla states, and the μ mix the errors with the excited ancilla states. Previous implementations of QECC kept the μ matrix diagonal and the γ, $\kappa = 0$, so that separate errors excited orthogonal ancilla states. For AQEC, we must check that each of the six eigenstates of the funnels have some non-zero projection along a state with excited ancilla so that population is not trapped in a funnel.

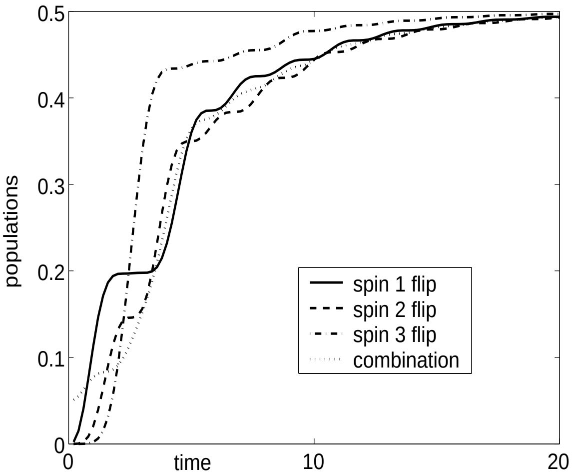
Eq. (8) shows some examples for H. All three sets properly repair spin-flip errors, but set (C), which is nearest in spirit to QECC, implements the most rapid repair. For the simulations, only the 14 total codeword and funnel states are used in the numerical simulations, so Γ is 196×196 in size. The matrix exponential routine of MATLAB [28] was used to produce $\exp(\Gamma\, t)$. The initial ρ is found by tracing the environment out from the initial error state, $|\Psi\rangle|e_0\rangle + I_{1,x}|\Psi\rangle|e_1\rangle + I_{2,x}|\Psi\rangle|e_2\rangle + I_{3,x}|\Psi\rangle|e_3\rangle$, where $|\Psi\rangle = (1/\sqrt{2})|000,00\rangle + (\exp(i\pi/3)/\sqrt{2})|111,00\rangle$. This state allows us to check whether the coherence phase is properly recovered. In general, the environmental overlaps $\langle e_n|e_m\rangle$ could be any complex numbers subject to $\sum_n \langle e_n|e_n\rangle = 1$ and $|\langle e_n|e_m\rangle|^2 \leq \langle e_n|e_n\rangle\langle e_m|e_m\rangle$.

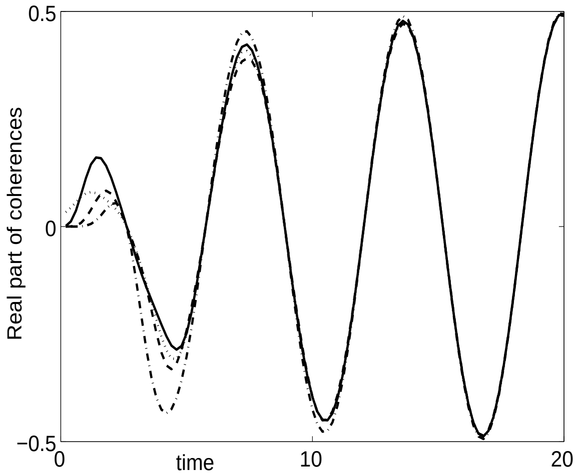
$$A \left(\begin{array}{c|ccc|ccc} 10 & 0 & 0 & 0 & 0 & 0 & 0 \\ \hline 0 & 2 & 1 & 0 & 1 & 0 & 0 \\ 0 & 1 & 2 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 2 & 0 & 0 & 0 \\ \hline 0 & 1 & 0 & 0 & 2 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 2 \end{array}\right) \quad B \left(\begin{array}{c|ccc|ccc} 10 & 0 & 0 & 0 & 0 & 0 & 0 \\ \hline 0 & 2 & 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 2 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 2 & 1 & 0 & 0 \\ \hline 0 & 0 & 0 & 1 & 2 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 & 2 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 & 2 \end{array}\right) \quad C \left(\begin{array}{c|ccc|ccc} 10 & 0 & 0 & 0 & 0 & 0 & 0 \\ \hline 0 & 2 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 2 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 2 & 0 & 0 & 1 \\ \hline 0 & 1 & 0 & 0 & 2 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 2 \end{array}\right) \tag{8}$$

Set Eq. (8,A) uses a single ancilla to correct the three independent spin-flip errors. Fig. (6) shows the recovery of the codeword populations and coherences for a spin-flip error at each S, and for a spin-flip error with a set of randomly chosen environmental overlaps, $\langle e_n | e_m \rangle$, as given in Eq. (9).

$$\begin{pmatrix} .10 & -.7+.2i & 0 & -.3-.3i \\ & .41 & .3+.7i & .4-.2i \\ & & .27 & .8+.3i \\ & & & .22 \end{pmatrix} \tag{9}$$

The H of Eq. (8,A) excites an ancilla only for the first spin-flip error. It repairs the other spin-flip error by mixing all the errors together. The numerical simulations shown in Fig. (6) show this process in detail. Note that if the all three spin-flips entangle the system with the environment, then the linear entropy of the system is initially non-zero.





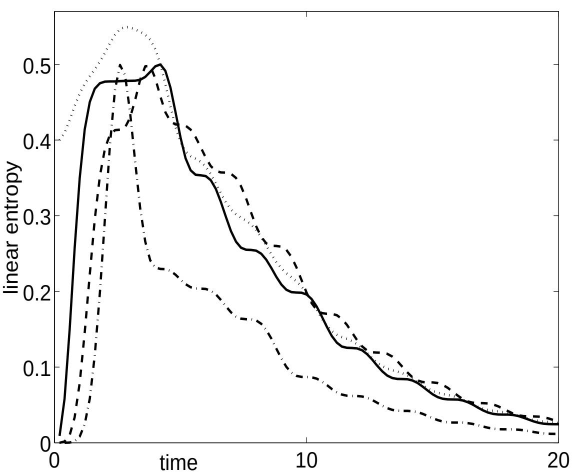


Fig. 6. Repair of the populations (upper left), coherences (upper right), and the linear entropy (lower), for errors in which each S alone flips, or there is a correlated flip of all the spins whose environmental overlaps are given by Eq. (9). All three graphs have the same legend. The repair process is a linear dynamics, so it also repairs correlated single spin-flips. Populations and coherences of both codewords follow identical paths. The parameter set is Eq. (8,A), but with the second codeword / funnel offset by $\Delta\omega = 1$ in order to show the oscillation of the coherences. The cooling rates were $c_1, c_2 = 1$.

Thus, AQEC can expell the information about which error occurred at the same time as the error is repaired. Eq. (8,B) is another example of this. It mixes together all the errors, and all the excited ancilla states. Because it does so symmetrically between the separate codewords, the errors are repaired. Fig. (7) shows the populations for all the funnel states after the first spin is flipped. It can be observed that all the states are transiently excited: the state vector "swirls around" in each funnel as the repair occurs.

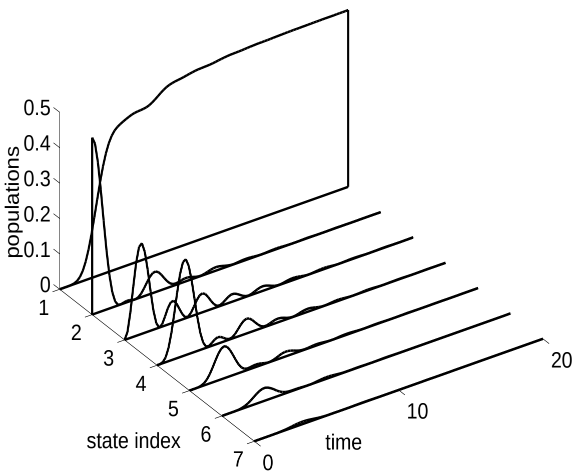


Fig. 7. The populations of the codewords (at back), of the three different spin-flip states (middle three), and of the three orthogonal excited ancilla states (forward three), during a repair after the first spin is flipped. The parameter set is Eq. (8,B), with cooling rates $c_1, c_2 = 1$. It is permissible under AQEC to mix together different errors during the repair, so long as the dynamics between the separate codeword / funnel subspaces is indistinguishable to the bath.

What happens when condition (**5**) is violated? There are two possibilities: excite the ancilla asymmetrically between the codewords, or have different dynamics between the two codeword / funnel subspaces. Fig. (8) shows the first case, for which Eq. (8,A) was used, but modified for the funnel surrounding $|111,00\rangle$ by setting $\mu_{11} = \mu_{12} = 1/\sqrt{2}$. The cooling rates were $c_1, c_2 = 1$, and the error was $I_{1,x}$. The partially orthogonal ancilla states do not allow Γ to return ρ to a pure state. In fact, if μ_{11} were set to zero for the second codeword, then there would be no element in Γ to transfer $|000,10\rangle\langle 111,01|$ to $|000,00\rangle\langle 111,00|$. The bath gains information about the system through excitation. The coherence asymptotically approaches $0.3530\exp(i\pi(0.3333))$, with correct phase but low magnitude.

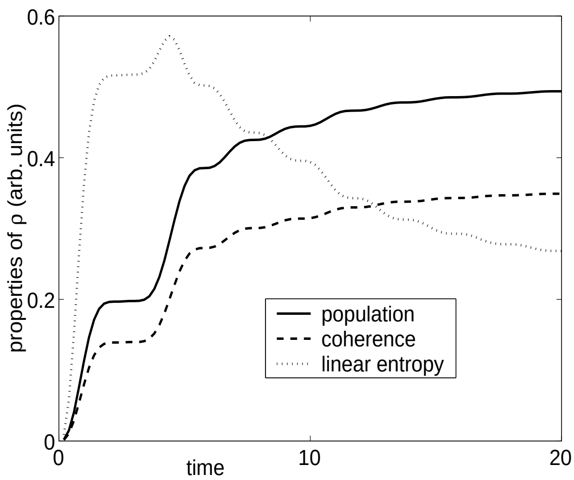


Fig. 8. A partial repair due to the use of partially orthogonal excited ancilla states, $|10\rangle$ and $(|10\rangle + |01\rangle)/\sqrt{2}$, between the two codewords. The populations (solid line) are repaired, but the coherence magnitudes are not (dashed line).

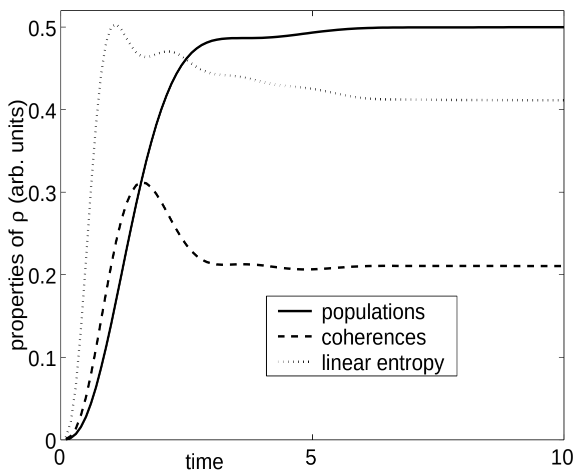


Fig. 9. A partial repair resulting from a dissimilar dynamics between the two codeword / funnel subspaces. The populations are repaired (solid line), but the coherence magnitudes are not (dashed lines).

The second possibility is shown in Fig. (9). Here, the ancilla are excited symmetrically, but the dynamics between the funnels is not equivalent, and coherence is again lost. The parameter set is Eq. (8,C), but μ_{11}=2 for the second funnel, so the mixing was more rapid. The cooling rates were $c_1, c_2 = 1$, and the error was $I_{1,x}$. Again, the population is repaired, but some coherence is lost.

5 A Proposed Test System.

We now give a physically realizable example of an AQEC system that implements Shor's three-qubit, majority code against spin-flip errors [1]. It can not repair phase-flip errors, so it is not suitable for a quantum computer. It serves to illustrate a method by which to find systems suitable for AQEC. It is also encouraging that a system can be found without making recourse to exotic interactions.

Our strategy is to find a system that obeys multiple conservation laws that the errors violate. Consider a system with an observable A such that $[H, A] = 0$, and an error E where $[A, E] = E$. The simultaneous eigenstates of H and A, $|\epsilon, a\rangle$, have the property that $AE|a\rangle = (a + 1)E|a\rangle$. Thus, choosing codewords with $a = 0$ and 2 gives rise to funnel states with $a = 1$ and 3. If, in addition, there is a unitary B such that $B|\epsilon, a\rangle = |\epsilon, a + 2\rangle$, then the funnel states can be mapped onto one another, and their dynamics are equivalent.

Let three spin 1/2 particles be lined up along the z axis, in a zero static magnetic field. They interact by point dipolar $D_{nm}(I_{n,x}I_{m,x} + I_{n,y}I_{m,y} - 2I_{n,z}I_{m,z})$ and exchange $J_{nm}(I_{n,x}I_{m,x} + I_{n,y}I_{m,y} + I_{n,z}I_{m,z})$ terms [15]. Dipolar interactions decrease with distance as r^{-3}, so for equally spaced spins, D_{12}=D_{23}=$8D_{13}$=ζ, where ζ can be as large as 0.1 cm^{-1} [29].

Assuming the dipolar interactions dominate, the level diagram of the spins is given in Fig. (10,A). The spins attempt to mutually align, resulting in ground states of $|000\rangle$ and $|111\rangle$. These are the codewords. A spin-flip error, $I_{n,x}$, is equivalent to rotating a spin by π about the x axis. Since this requires work against the dipolar field, dissipation can repair these errors. The funnels come from the conservation of the spin angular momentum about the z axis, $\sum_n I_{n,z}$, which has eigenvalues denoted as m_z. An error changes $m_z \to m_z \pm 1$. The codewords have $m_z = \pm 3/2$. The funnel surrounding $|000\rangle$ (levels A-C of Fig. (10,A)) has $m_z = -1/2$, and the funnel surrounding $|111\rangle$ has $m_z = +1/2$ (levels D-F).

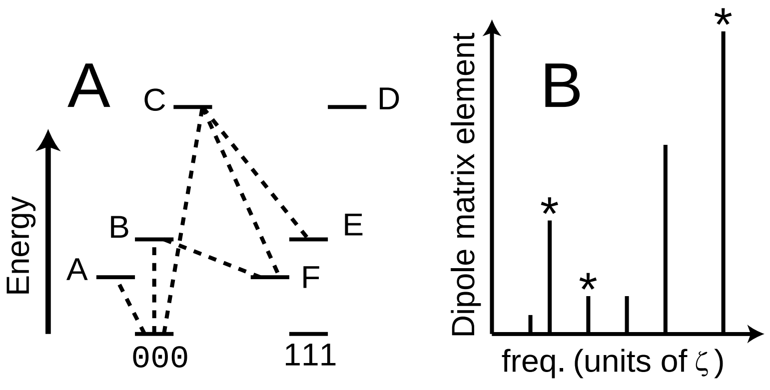


Fig. 10. (**A**) The level diagram for the three spin system. The ground states are the codewords. States A-C with $m_z = -1/2$ form the funnel for $|000\rangle$. The dashed lines show the dipole-allowed transitions for the first funnel, with symmetric transitions for the second funnel. (**B**) The spectrum of dipole-allowed transitions. The starred lines represent the funnel to codeword transitions that should be cooled.

Spontaneous emission of a photon with an x polarized B field will symmetrically de-excite the degenerate funnels. Thus, we could use photons as the ancilla for this system. There are several advantages to this approach. First, it is difficult to selectively cool a single spin, but it is easy to cool an electromagnetic resonator. Second, the validity of the Markov approximation for a damped resonator mode is better understood

[9, 16]. A y-polarized B field will anti-symmetrically de-excite the funnels, so an error $I_{n,y}$ will be "repaired" to the phase-flip error $I_{n,z}$. This phase-flip error can not be repaired, because it requires no work to rotate a spin about the z axis for this system.

There are two last difficulties to overcome. The dashed lines of Fig. (10,A) show the dipole-allowed transitions for the $m_z = -1/2$ states. Their strengths are proportional to matrix elements of the operator $\sum_n I_{n,x}$. Point dipolar interactions alone are not sufficient for AQEC, because $\langle B|\sum_n I_{n,x}|000\rangle$ is always zero, no matter how the spins are positioned along the z axis. This unwanted symmetry is broken by setting $J_{12} = J_{13} = 0$ and $J_{23} = 0.2\zeta$. Actually, any $0 < |J_{23}| \leq 0.5\zeta$ will work. Using the above parameters gives rise to the spectrum of Fig. (10,B). The second difficulty is that, besides spontaneous emission of y-polarized B-field photons, there are also transitions between the funnels. We wish to cool only the starred transitions (at 0.64ζ, 1.03ζ, and 2.39ζ), but not the un-starred ones (at 0.39ζ, 1.36ζ and 1.75ζ).

This can be achieved by placing the spins at the center of a resonator whose modes are only resonant with the starred transitions. Consider a rectangular, conducting cavity of linear dimensions a, b, and d. Resonances are indexed as transverse electric (TE$_{mnp}$, where $n + m > 0$, $p > 0$) and transverse magnetic (TM$_{mnp}$, where $n, m > 0$, $p \geq 0$) modes [30], with frequencies $\omega_{mnp} = \sqrt{(m/a)^2 + (n/b)^2 + (p/d)^2}$ in units of cm^{-1} if the cavity lengths are in cm. Each mode produces either a linearly polarized electric or magnetic field, or no field, at the center. One can invert the above to find that a resonator with dimensions a=$2.32/\zeta$, b=$0.87/\zeta$, and d=$4.28/\zeta$, has TE$_{102}$, TE$_{104}$, and TE$_{122}$ modes resonant with the starred transitions. Each mode has an x polarized B field at the spins. Another 29 modes exist with $\omega < 2.5\zeta$. Of those that produce B fields at the spins, the nearest to a funnel-funnel transition is TE$_{302}$, which is offset by 0.018ζ from the C-E transition. A resonator Q $\gg 76$ is required to suppress emission of this transition. For $\zeta \approx 0.1$ cm^{-1}, microwave resonators can achieve this goal. This larger ζ is also desirable because the cold bath must satisfy $T \ll (hc/k)\zeta \approx 0.1$ K, a not outrageous requirement.

6 Discussion

AQEC borrows the same structure for storing information as in QECC, but implements the error correction in a different way. In NMR terminology, the qubit of the above system is hidden in the triple quantum coherence of the spins. The novel aspect is that dissipation can be used to directly repair not only the codeword populations, but also the coherences. The criteria for this is simply summed up by demanding that excitation, and environmental entanglements, be expelled from the codewords in a symmetric manner.

Especially interesting is the possibility that an AQEC qubit exists that can protect against both spin- and phase-flip errors. Exchange interactions may prove more useful in this regard. Being isotropic interactions, they resist the rotation of a spin about any axis. One difficulty with using only exchange interactions, is that dipole-allowed transitions vanish, so a symmetric de-excitation of all the funnels by photons becomes problematic. Another open question is how AQEC behaves when it is scaled up to large numbers of codewords. It is, however, helpful to contemplate an error correction scheme that requires no additional burden to the programmer.

7 Conclusions

Conditions are given by which the dissipative evolution of a system, coupled to a cold Markovian bath, can be used to implement automatic quantum error correction. The new condition, necessary to repair codeword coherences, requires a symmetric de-excitation of separate codewords, and an equivalent dynamics between the different codeword / funnel subspaces. They resemble the conditions of phase-matching in nonlinear optics. A test case, that of Shor's majority-code against spin-flip errors [1], is proposed. It utilizes well known dipolar and exchange interactions between spins, and dipole-allowed transitions with the modes of a resonator.

7.1 Acknowledgements

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